Int. J. Solids Structures Vol. 29, No. 6, pp. 721–733, 1992
Printed in Great Britain.

0020-7683/92 \$5.00+.00

ON THE PERTURBATION ANALYSIS OF INTERACTIVE BUCKLING IN NEARLY SYMMETRIC STRUCTURES

A. LUONGO
Dipartimento di Ingegneria delle Strutture, Universita' di L'Aquila, Monteluco Roio, 67040 L'Aquila, Italia

and

M. PIGNATARO
Dipartimento di Ingegneria Strutturale e Geotecnica, Universita' di Roma "La Sapienza", 00184 Roma, Italia

(*Received* 10 *October* 1990; *in revised form* 2 *June* 1991)

Abstract—Different perturbation methods for the analysis of non-linear interaction between simultaneous buckling modes of nearly symmetric structures are discussed. First, the perturbation method employed by Budiansky for a single buckling mode, is extended to consider modes interaction of a perfect structure, by determining both the slope and the curvature of the bifurcated paths. It is shown that the solution diverges, when a properly defined parameter which characterizes the asymmetry of the structure approaches zero, thus preventing to recover results of symmetric systems. A modified perturbation method which permits to surmount this drawback is then suggested; this method applies only to a class of structures and furnishes asymptotic series valid in a wide region around bifurcation. The two methods are applied to investigate the post-buckling behavior of a two-degree-of-freedom system. Finally, a novel perturbation method which follows to some extent the lines of the Galerkin method and is particularly powerful in the investigation of nearly symmetric systems is presented.

1. INTRODUCTION

Non-linear problems, both in statics and dynamics, are often treated with perturbation techniques which have the advantage of being easy to apply and able to furnish parametric solutions of the problems (Koiter, 1945; Budiansky, 1974; Nayfeh, 1973). The algorithm, when applied, for instance, to an equation of the type

$$L(u;\lambda)+c_2u^2+c_3u^3+\cdots=0, \tag{1}$$

where L is a linear operator and c_2, c_3 are constants, consists in expressing the state variable u and the control parameter λ as a power series in terms of a perturbation parameter ξ

$$\begin{aligned} u &= \xi u_1+\tfrac{1}{2}\xi^2u_2+\cdots \\ \lambda &= \lambda_c+\xi\lambda_1+\tfrac{1}{2}\xi^2\lambda_2+\cdots \end{aligned} \tag{2}$$

and in determining the coefficients of the series expansion through the solution of a sequence of linear perturbation equations.

Usually, the series expansion is truncated at the first term which is different from zero. For instance, in a bifurcation problem where λ denotes the load, the expansion is truncated at the linear terms if $\lambda_1 \neq 0$, or at the quadratic terms if $\lambda_1 = 0$ and $\lambda_2 \neq 0$. The equilibrium

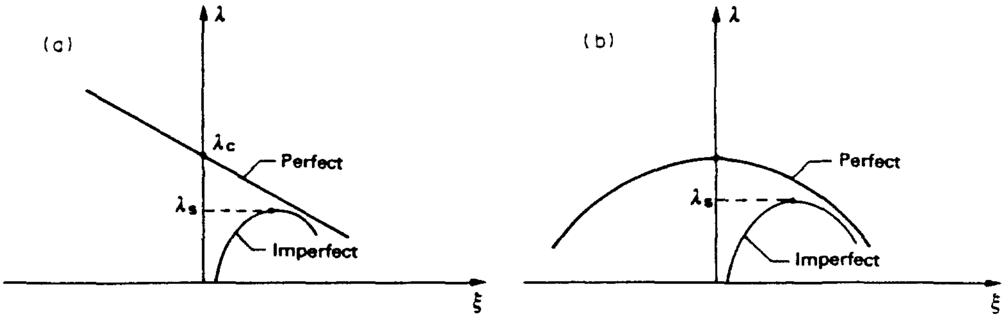

Fig. 1. Equilibrium paths: (a) asymmetric system ($\lambda_1 \neq 0$); (b) symmetric system ($\lambda_1 = 0$, $\lambda_2 \neq 0$).

paths of perfect systems shown in Fig. 1 are correspondingly obtained which are a straight line (asymmetric structures) or a parabola (symmetric structures); for imperfect systems a snapping load λ_s is present.

There exists, however, a wide class of problems in which λ_1 is so small that, for values of ξ in the domain of interest, second order terms cannot be neglected in that they strongly characterize the mechanical behavior of the systems. Their evaluation requires to extend the analysis one step further at the cost of a remarkable increase of the computational effort. Systems obeying this type of behavior are here called *nearly symmetric*.

Nearly symmetric systems have deserved little attention in the literature. A number of references regarding static and dynamic buckling problems can be found in Elishakoff (1980) where the post-critical behavior of a one-degree-of-freedom system (Fig. 2a) with quadratic and cubic non-linearities has been analyzed. The non-dimensional parameter $\chi = k_2 l/k_1$ gives a measure of the asymmetry of the system which is proportional to the slope of the equilibrium path at bifurcation (Fig. 2b). For small values of χ the system is nearly symmetric. The equilibrium path of the perfect system shows that, for displacements $\xi > 0$, the load λ decreases by reaching a minimum and then increases again ($\lambda_1 < 0$, $\lambda_2 > 0$) within a domain ξ inside which the asymptotic solution is valid. Consequently, the equilibrium paths of the imperfect system exhibit limit points only for sufficiently small initial imperfections (system sensitive to initial imperfections); for increasing values of the imperfections amplitude $\bar{\xi}$ the limit points disappear (system insensitive to initial imperfections). In conclusion, the mechanical behavior of the system is asymmetric and is governed by quadratic non-linearities in a small neighborhood of the origin while it is symmetric and is governed by cubic non-linearities outside this neighborhood. Therefore, a perturbation analysis truncated at the first order furnishes results which are correct only around the bifurcation point and are wrong if extrapolated in a wider region.

The system illustrated is characterized by having a single critical mode. However, many

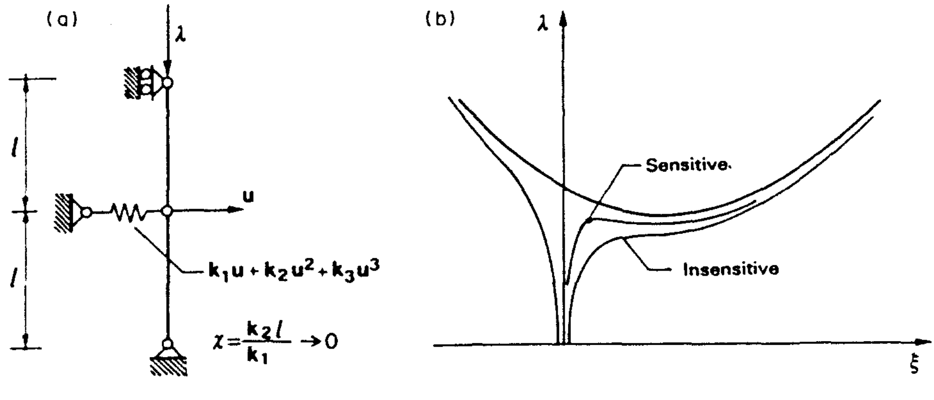

Fig. 2. Nearly symmetric system: (a) one d.o.f. model; (b) equilibrium paths.

problems which are encountered in practice, manifest a number of buckling modes by correspondence with the same value of the critical load λ_c (interactive buckling).

Interactive problems have been extensively investigated in the past years. The interest has in particular been focused on the analysis of thin-walled members under compression which may undergo local or overall buckling [see for example the papers by Byskov and Hutchinson (1979), Sridharan and Benito (1984), Bradford and Hancock (1984), Benito and Sridharan (1985), Sridharan and Ali (1985), Pignataro *et al.* (1985), Pignataro and Luongo (1987), Luongo and Pignataro (1989) and Byskov *et al.* (1989)] but little attention has been devoted to nearly symmetric structures.

In this paper we wish to analyze the post-buckling behavior of nearly-symmetric elastic systems which exhibit several buckling modes and show that some perturbation techniques lead to inaccurate results.

2. PERTURBATION ANALYSIS OF SIMULTANEOUS BUCKLING MODES

The equilibrium paths of an elastic structure subjected to conservative loads which exhibits a number of simultaneous buckling modes are determined. First, the standard perturbation analysis is applied by following the Budiansky (1974) formulation where the series expansion is carried out one step further; then a modified perturbation method is presented which is applicable to a particular class of systems. A brief sketch of the two procedures is presented here; details can be found in Pignataro and Luongo (1988) and Luongo and Pignataro (1988).

2.1. *Standard perturbation method*

Let us consider a hyperelastic body system subjected to conservative loads characterized by the total potential energy functional $\Phi[w;\lambda]$, where w is the displacement field and λ a parameter governing the external force field acting on it. The equilibrium condition is obtained by requiring the functional $\Phi[w;\lambda]$ to be stationary with respect to kinematically admissible displacement fields, that is

$$\Phi'[w;\lambda]\delta w = 0 \quad \forall\, \delta w, \tag{3}$$

where a prime denotes Fréchet differentiation with respect to w.

In buckling problems it is assumed that at a certain critical value λ_c of the load factor λ, the state w_c belongs to two different equilibrium paths: the fundamental one $w_0(\lambda)$, which is taken to be known, and the bifurcated path $w(\lambda)$. In general, the fundamental path is no easier to find than any other path, and in an analytical sense it is known *a priori* only for simple cases. However, in many problems, an approximate description of the fundamental path is sufficient for an adequate estimate of the post-critical behavior.

By introducing the differential state variable $u(\lambda)$, the bifurcated path can be described as

$$w(\lambda) = w_0(\lambda) + u(\lambda). \tag{4}$$

It is usually convenient to express the function $u(\lambda)$ using the parametric relations $u = u(\xi)$, $\lambda = \lambda(\xi)$ which are assumed to be regular. These relations can then be expressed through the series expansion (2), where $\xi = 0$ corresponds to bifurcation. By substituting eqn (4) into (3), by expanding this equation in terms of u and λ starting from $u = 0$ and $\lambda = \lambda_c$, respectively, and by using relations (2) we get

$$\xi\{\Phi_c''u_1\}\delta u + \tfrac{1}{2}\xi^2\{\Phi_c''u_2 + 2\lambda_1\dot{\Phi}_c''u_1 + \Phi_c'''u_1^2\}\delta u + \tfrac{1}{6}\xi^3\{\cdots\}\delta u = 0. \tag{5}$$

By equating separately to zero all terms with the same power of ξ, the following perturbation equations are obtained:

$$\Phi_c'' u_1 \delta u = 0$$
$$\Phi_c'' u_2 \delta u = -\{2\lambda_1 \dot{\Phi}_c'' u_1 + \Phi_c''' u_1^2\}\delta u$$
$$\Phi_c'' u_3 \delta u = -3\{\lambda_1 \dot{\Phi}_c'' u_2 + \lambda_2 \dot{\Phi}_c'' u_1 + \lambda_1^2 \ddot{\Phi}_c'' u_1 + \Phi_c''' u_1 u_2 + \lambda_1 \dot{\Phi}_c''' u_1^2 + \tfrac{1}{3}\Phi_c'''' u_1^3\}\delta u, \tag{6}$$

where

$$\Phi_c'' = \Phi''[w_0(\lambda_c);\lambda_c]$$
$$\dot{\Phi}_c'' = \left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi''[w_0(\lambda);\lambda]\right]_{\lambda=\lambda_c}$$
$$= \left[\Phi'''[w_0(\lambda);\lambda]\frac{\mathrm{d}}{\mathrm{d}\lambda}w_0(\lambda) + \frac{\partial}{\partial\lambda}\Phi''[w_0(\lambda);\lambda]\right]_{\lambda=\lambda_c}. \tag{7}$$

Analogous positions hold for higher order differentiations. Note that the procedure breaks down when bifurcation occurs at a limit load ($\mathrm{d}w_0/\mathrm{d}\lambda = \infty$) and therefore this case will be excluded in the sequel.

Equation (6a) is an eigenvalue problem which is assumed to admit the multiple eigenvalue λ_c and the m eigenfunctions v_i. The first order displacements field may therefore be expressed as a linear combination with arbitrary coefficients μ_i of the m independent solutions

$$u_1 = \mu_i v_i \quad (i = 1, 2, \ldots, m). \tag{8}$$

After replacing eqn (8) into (6b), coefficients μ_i are determined by imposing on the second hand member to be orthogonal to all eigenfunctions v_i (Fredholm). This leads to the set of m equations

$$A_{ijk}\mu_j\mu_k + \lambda_1 B_{ik}\mu_k = 0 \quad (i, j, k = 1, 2, \ldots, m), \tag{9}$$

which are quadratic in μ_i and bilinear in λ_1, μ_i where

$$A_{ijk} = \Phi_c''' v_i v_j v_k, \quad B_{ik} = 2\dot{\Phi}_c'' v_i v_k. \tag{10}$$

By adding to (9) a normalization condition such as, for instance, $\mu_i\mu_i = 1$, a set of p solutions is obtained where, according to the Bezout theorem, p is real at the most equal to $2^m - 1$ and at least equal to one.

By solving eqn (6b), the second order displacements field $u_2 = v_p + \beta_i v_i$ is obtained, where v_p is a particular integral and β_i arbitrary constants which are determined with λ_2 by imposing the Fredholm condition on eqn (6c). The following set of linear equations is obtained:

$$\left[\begin{array}{c|c} 2A_{ijk}\mu_j + \lambda_1 B_{ik} & B_{ik}\mu_i \\ \hline \mu_i & 0 \end{array}\right]\left\{\begin{array}{c}\beta_i \\ \hline \lambda_2\end{array}\right\} = \left\{\begin{array}{c} f_k(\mu_i, \lambda_1) \\ \hline g(\mu_i)\end{array}\right\}, \tag{11}$$

where f_k and g are known functions and the last equation is a normalization condition expressing the orthogonality between u_1 and u_2. Equations (11) are solved for each p-tuple (μ_i, λ_1) and furnish the second order coefficients β_i and λ_2 for each bifurcated path.

If the system is symmetric, then all coefficients A_{ijk} vanish and eqns (9) furnish $\lambda_1 = 0$.

Coefficients μ_i remain undetermined at this level and are evaluated with λ_2 from the solvability conditions of eqns (6c) which read

$$2A_{ijkl}\mu_i\mu_j\mu_l+\lambda_2 B_{ik}\mu_i = 0 \quad (i,j,k,l = 1,2,\ldots,m), \tag{12}$$

to which the condition $\mu_i\mu_i = 1$ has to be added. In eqns (12) it is

$$A_{ijkl} = \Phi_c'''v_i v_{jk} v_l + \tfrac{1}{3}\Phi_c''''v_i v_j v_k v_l, \tag{13}$$

being $u_2 = \mu_j\mu_k v_{jk}$ a particular solution of eqn (6b). The coefficients β_i which are undetermined at this level may be set equal to zero. Equations (12) furnish $1 \leqslant q \leqslant 3^m - 1$ real solutions. When initial imperfections are present, eqns (9), (11) and (12) are corrected by adding an extra term which accounts for the amplitude imperfections ξ and for the corresponding shape [see Pignataro and Luongo (1988)].

The case which is of interest arises when the system is nearly symmetric. In this case it is $A_{ijk} = O(\chi)$ where χ is a small parameter and therefore, from eqns (9), it is $\lambda_1 = O(\chi)$. The matrix of system (11) is thus ill-conditioned and the second order coefficients β_i and λ_2 tend to infinity, as they are of order $O(\chi^{-1})$. Consequently, results furnished by the series expansion (2) are valid in a very small neighborhood of the bifurcation point only; indeed, for $\xi = O(\chi)$, second order terms are no longer a small correction of the first order terms and the asymptotic series expansion is no more uniformly valid. In addition, for $\chi \to 0$ the solution of the asymmetric system diverges, thus preventing the recovery of the solution relative to the symmetric case which is governed by the set of cubic eqns (12). This implies the loss of the main feature of the perturbation method in that parametric solutions can no longer be obtained.

It is worthwhile to observe that this drawback arises only in interactive buckling. If there is only one buckling mode, then $\mu_1 = 1$ and eqn (9) furnishes $\lambda_1 = -A_{111}/B_{11}$. Equations (11) become

$$\begin{bmatrix} A_{111} & B_{11} \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ g \end{Bmatrix}, \tag{14}$$

which are well conditioned even if $A_{111} \to 0$. Besides, by expliciting f_1, one can see that the solution approaches that of the symmetric system for $\chi \to 0$ (Pignataro and Luongo, 1988). The reason for the different behavior of the solutions of the two problems corresponding to $m > 1$ and $m = 1$ lies in the fact that in the first case, in contrast with single buckling problems, the number and the directions of the bifurcated paths are not known.

2.2. *Modified perturbation method*

From previous discussions on the results of the standard method, it appears desirable to formulate a procedure for nearly symmetric systems which allows one to obtain a solution valid in a wider region around bifurcation, from which the symmetric solution is consistently recovered when $\chi \to 0$. The following preliminary considerations may guide our reasoning in finding a correct approach for the solution of the problem:

(a) the post-buckling behavior is always governed by third order terms of the energy in a suitably chosen region around bifurcation, the amplitude of which depends on χ;

(b) far from bifurcation fourth order terms become dominant within the domain of interest;

(c) there exists an intermediate region inside which third and fourth order terms of the energy are comparable and therefore both are necessary to describe the mechanical behavior of the system. This implies that in the asymptotic procedure they must appear in the same order perturbation equation.

Our aim is achieved if we succeed in describing the structure behavior inside the transition region. To this end let us assume that all third order terms are small, of order χ, that is

$$\Phi_\xi''' = \chi\tilde{\Phi}_\xi''', \quad \dot{\Phi}_\xi''' = \chi\dot{\tilde{\Phi}}_\xi''' \tag{15}$$

where we define

$$\chi = \kappa\xi. \tag{16}$$

The standard perturbation eqns (6b) are therefore modified by shifting third order terms to the next order equation. By following the same steps as in the previous case, $\lambda_1 = 0$, $u_2 = 0$ are found and the Fredholm conditions are rewritten as

$$2A_{ijkl}\mu_i\mu_j\mu_l + \lambda_2 B_{ik}\mu_i + 2\kappa\tilde{A}_{ijk}\mu_i\mu_j = 0, \tag{17}$$

where $\tilde{A}_{ijk} = \tilde{\Phi}_\xi''' v_i v_j v_k$. Note that eqns (17) are the same as eqns (12) relative to the symmetric system with an extra term accounting for the weak asymmetry of the structure. In the following we shall refer to eqns (17) as *modified perturbation equations*. These equations, with the normalization condition $\mu_i\mu_i = 1$, furnish μ_i and λ_2 as a function of the parameter κ and therefore of ξ. The solution is then written in the form

$$\lambda = \lambda_c + \tfrac{1}{2}\lambda_2(\xi)\xi^2, \quad u = \mu_i(\xi)v_i\xi. \tag{18}$$

Faced with imperfections it is sufficient to add an extra term to eqns (17) (Luongo and Pignataro, 1988).

Equations (17) describe the bifurcated paths in a wide domain around bifurcation. They are suited to a numerical solution as well as to an asymptotic solution for large values of κ (i.e. in accordance with eqn (16), in the neighborhood of bifurcation) and for small values of κ (i.e. far from bifurcation). In this way the solution can be approximated both by an extrapolation from the origin, which is valid in a small neighborhood of it and by an extrapolation starting from a point far away; the two solutions have then to be matched.

Around bifurcation it is seen that, as $\mu_i = O(1)$, for large values of κ we have $\lambda_2 = O(\kappa)$. The problem is therefore governed in this case by the second and third term in eqns (17) which characterize the asymmetric behavior. The solution can be determined as a perturbation of that corresponding to $\kappa = \infty$; performing a number of straightforward steps, the standard method results relative to the asymmetric structure are recovered.

Far from bifurcation κ assumes small values. In this case the problem is governed by the first two terms of the modified perturbation equation which describe the symmetric behavior of the structure. An asymptotic solution can therefore be obtained as a perturbation of that corresponding to $\kappa = 0$. By letting

$$\lambda_2 = \lambda_2^0 + \kappa\lambda_2^* + \cdots, \quad \mu_i = \mu_i^0 + \kappa\mu_i^* + \cdots, \tag{19}$$

the following solution is obtained:

$$\lambda = \lambda_c + \tfrac{1}{2}\lambda_2^0\xi^2 + \tfrac{1}{2}\chi\lambda_2^*\xi + \cdots, \quad u = (\mu_i^0\xi + \chi\mu_i^*)v_i + \cdots, \tag{20}$$

where μ_i^0, λ_2^0 are solutions of the zeroth order perturbation equations which coincide with eqns (12) and μ_i^*, λ_2^* are obtained by solving a linear problem. Equation (20b) furnishes straight lines in the plane of the displacements u which are parallel to those relative to the symmetric system because of the presence of the constant term $\chi\mu_i^*$. In the limit $\chi \to 0$ for which the standard procedure fails, the symmetric solution is obtained.

We may conclude that, in the neighborhood of the bifurcation, the structure behavior is described by paths which are near to those of the asymmetric system and successively come close to the equilibrium lines of the symmetric structure by remaining far away by a distance of order χ.

It is worth emphasizing that the second order displacements field u_2 is zero as a consequence of the assumption that the third order terms of the energy are small for *any displacements field*. Indeed, in the most common cases, the cubic terms of the energy are small only if displacements coincide with the buckling modes. This renders the modified perturbation method applicable only to a restricted class of problems.

3. ILLUSTRATIVE PROBLEM

The theory has been applied to investigate the post-critical behavior of the two d.o.f. system illustrated in Fig. 3a (Luongo and Pignataro, 1988). Let φ and ϑ be the Lagrangian parameters of the system measured as shown in Fig. 3b. The total potential energy can be written as

$$\Phi = \tfrac{1}{2}k_1\varphi^2 + \tfrac{1}{2}k_2\vartheta^2 + \tfrac{1}{2}k_3 l^2\Delta^2 - Nw, \tag{21}$$

where k_1, k_2, k_3 are spring constants, N is the vertical load, w the vertical displacement of its point of application and Δ the stretching of the extensional spring. From kinematics, the following non-linear relations are obtained:

$$\Delta = \sqrt{(1+\sin\vartheta)^2 + \sin^2\varphi} \tag{22}$$

$$w = l(1 - \sqrt{1 - \sin^2\varphi - \sin^2\vartheta}). \tag{23}$$

If initial imperfections $\bar{\varphi}$, $\bar{\vartheta}$ are present, the energy (21) is modified as

$$\Phi = \tfrac{1}{2}k_1(\varphi - \bar{\varphi})^2 + \tfrac{1}{2}k_2(\vartheta - \bar{\vartheta})^2 + \tfrac{1}{2}k_3 l^2(\Delta - \bar{\Delta})^2 - N(w - \bar{w}), \tag{24}$$

where $\bar{\Delta}$ and $\bar{w}$ are obtained from (22) and (23) by replacing φ, ϑ with $\bar{\varphi}$, $\bar{\vartheta}$.

By taking the series expansion of eqn (24) up to fourth order terms in φ and ϑ and retaining only the bilinear terms $\varphi\bar{\varphi}$, $\vartheta\bar{\vartheta}$ in the initial imperfections we obtain

$$\Phi = \tfrac{1}{2}k_1[\varphi^2 + \vartheta^2 + \chi(\varphi^2\vartheta + \tfrac{1}{4}\varphi^4 - \tfrac{1}{3}\vartheta^4) - \lambda(\varphi^2 + \vartheta^2 - \tfrac{1}{12}\varphi^4 - \tfrac{1}{12}\vartheta^4 + \tfrac{1}{2}\varphi^2\vartheta^2)] - k_1(\varphi\bar{\varphi} + \vartheta\bar{\vartheta}). \tag{25}$$

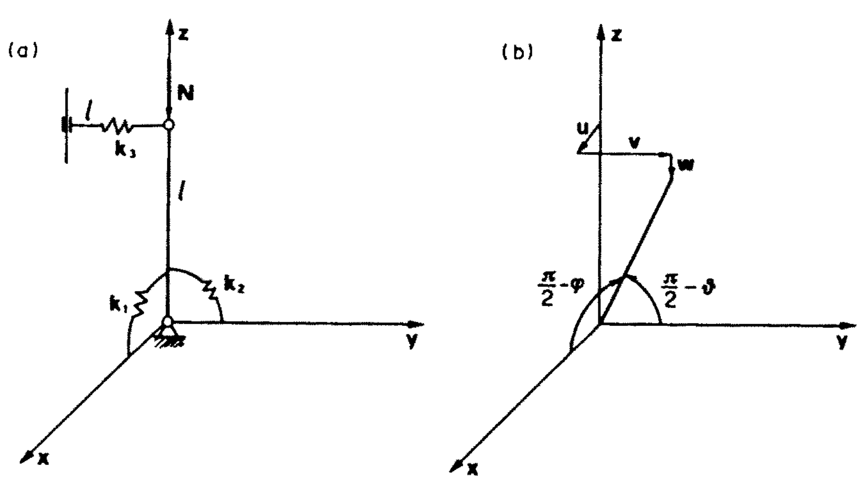

Fig. 3. Two d.o.f. model: (a) reference configuration; (b) varied configuration.

In eqn (25) $k_1 = k_2 + l^2 k_3$ has been taken in order to make the two critical loads to coincide and the following non-dimensional parameters have been introduced:

$$\chi = l^2 k_3 / k_1, \quad \lambda = Nl/k_1. \tag{26}$$

Note that $\Phi[-\varphi, \vartheta] = \Phi[\varphi, \vartheta]$ whereas $\Phi[\varphi, -\vartheta] \neq \Phi[\varphi, \vartheta]$. The asymmetry of the model is due to the extensional spring which is responsible for the cubic terms of the energy proportional to the factor χ $(0 \leqslant \chi \leqslant 1)$. A family of these models is examined, characterized by different values of χ, by first applying the standard method and then the modified method.

By applying the standard perturbation method (SPM) it is found that the asymmetric perfect system $(\chi \neq 0)$ exhibits $p = 3$ post-buckling equilibrium paths depicted in Fig. 4a in the plane of the Lagrangian parameters:

$$\mathscr{P}_1^*, \mathscr{P}_2^* : \begin{cases} \varphi = \left(\dfrac{2}{3}\right)^{1/2} \xi \pm \left(\dfrac{1}{2}\right)^{1/2} \left(\dfrac{2}{27}\dfrac{1}{\chi} + \dfrac{5}{27}\right)\xi^2 \\ \vartheta = \pm\left(\dfrac{1}{3}\right)^{1/2} \xi - \left(\dfrac{2}{27}\dfrac{1}{\chi} + \dfrac{5}{27}\right)\xi^2 \\ \lambda = 1 \pm \left(\dfrac{1}{3}\right)^{1/2} \chi\xi + \dfrac{1}{54}(8\chi - 7)\xi^2 \end{cases} \tag{27}$$

$$\mathscr{P}_3^* : \varphi = 0, \quad \vartheta = \xi, \quad \lambda = 1 + \tfrac{1}{6}(1 - 4\chi)\xi^2.$$

In the same figure curves $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ are the linear approximations of curves $\mathscr{P}_1^*, \mathscr{P}_2^*, \mathscr{P}_3^*$.

The symmetric system $(\chi = 0)$ has instead $q = 4$ equilibrium paths represented in Fig. 4b

$$\begin{aligned} &\mathscr{S}_{1,2} : \varphi = (1/2)^{1/2}\xi, && \vartheta = \pm(1/2)^{1/2}\xi, && \lambda = 1 - (1/6)\xi^2 \\ &\mathscr{S}_3 : \varphi = \xi, && \vartheta = 0, && \lambda = 1 + (1/6)\xi^2 \\ &\mathscr{S}_4 : \varphi = 0, && \vartheta = \xi, && \lambda = 1 + (1/6)\xi^2. \end{aligned} \tag{28}$$

Equations (27) and (28) are determined by solving eqns (9), (11) and (12), respectively.

Note that the asymmetric and symmetric systems exhibit a different number of bifurcated paths. In addition the solution (27) is unable to represent the behavior of the symmetric system for $\chi \to 0$ since second order terms diverge and therefore its validity for small values of χ is limited to a small neighborhood of the bifurcation point.

The modified perturbation method (MPM) has been successively utilized. The relevant modified perturbation equations read

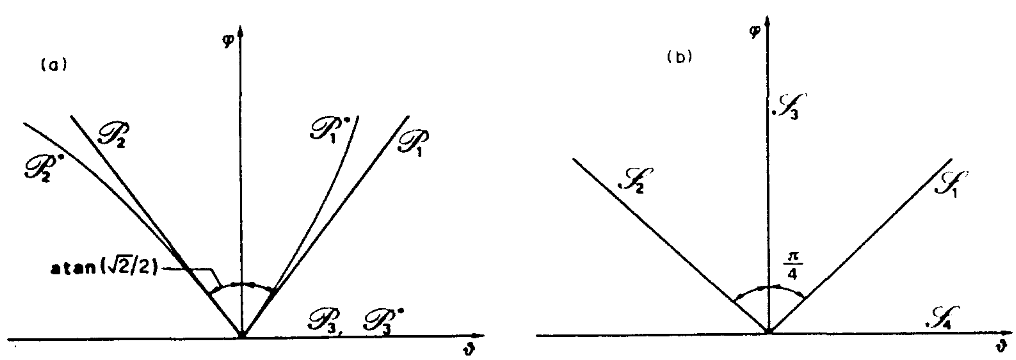

Fig. 4. Bifurcated paths: (a) asymmetric system $(\chi \neq 0)$; (b) symmetric system $(\chi = 0)$.

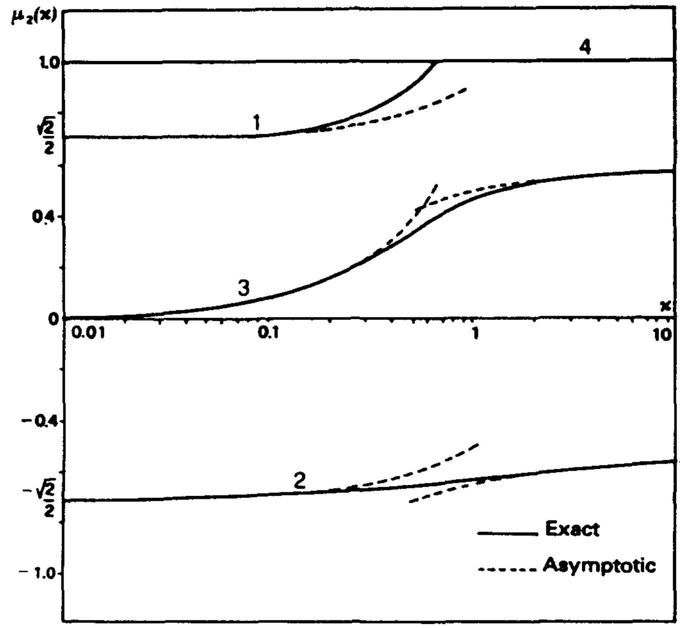

Fig. 5. Solutions $\mu_2(\kappa)$ to the modified perturbation equations.

$$
\begin{aligned}
&\tfrac{2}{3}\mu_1^3-2\mu_1\mu_2^2-2\lambda_2\mu_1+4\kappa\mu_1\mu_2 = 0\\
&\tfrac{2}{3}\mu_2^3-2\mu_1^2\mu_2-2\lambda_2\mu_2+2\kappa\mu_1^2 = 0\\
&\mu_1^2+\mu_2^2 = 1,
\end{aligned}
\tag{29}
$$

whose solutions furnish $\mu_1(\kappa)$ and $\mu_2(\kappa)$. Four solutions are obtained for $\mu_2(\kappa)$ and represented in Fig. 5 in semi-logarithmic scale. It is seen that for large values of κ, i.e. according to eqn (16), for small values of ξ, three solutions corresponding to those of the standard method for asymmetric systems are obtained; for small values of κ (large values of ξ), the four solutions furnished by the standard method for symmetric systems are recovered. If the non-linear equations (29) are solved asymptotically for large or small values of κ, the dashed curves shown in Fig. 5 are obtained. They are a good approximation of the exact solution with the exception of a region around $\kappa = 1$, i.e. $\xi = \chi$.

The situation is clearer if the equilibrium paths $\mathscr{R}_i$ $(i = 1,\ldots,4)$, corresponding to the solutions 1–4 in Fig. 5, are plotted in the φ, ϑ plane and compared with the $\mathscr{P}_i$ and $\mathscr{S}_i$ curves (Fig. 6). There are three paths $\mathscr{R}_i$ starting from the bifurcation point whose tangents coincide with curves $\mathscr{P}_i$; however they rapidly change direction and become parallel to

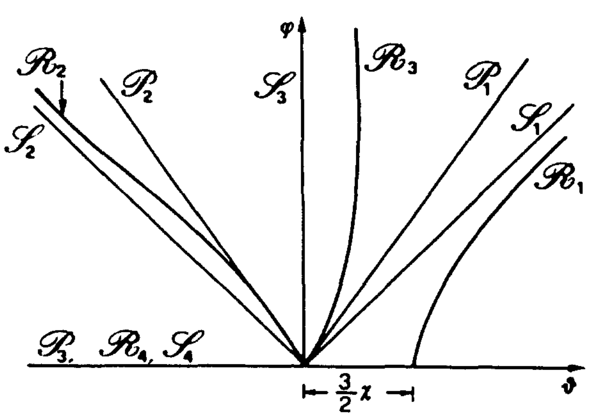

Fig. 6. Bifurcated paths by modified (MPM) and standard (SPM) perturbation methods.

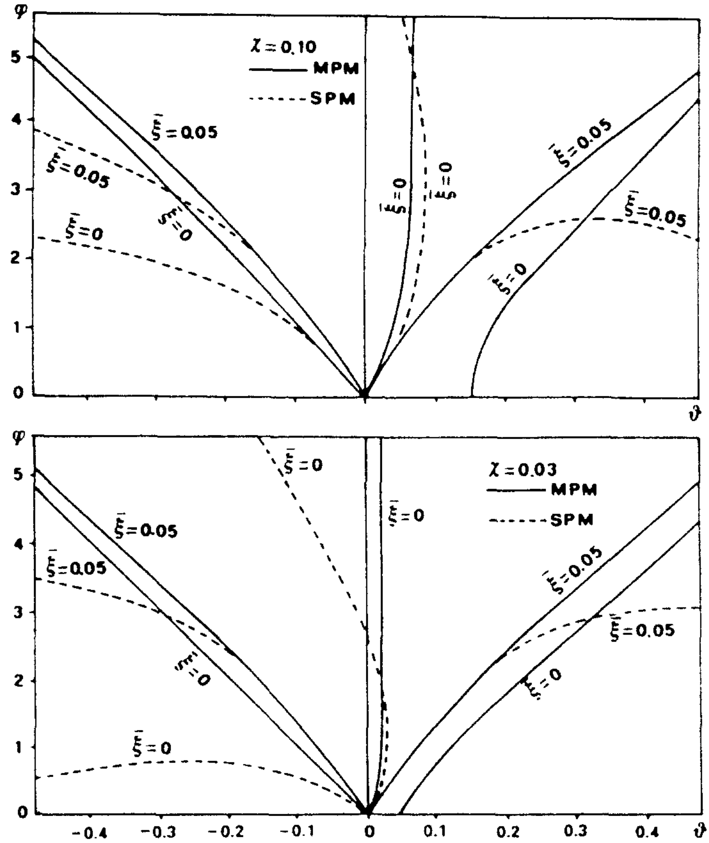

Fig. 7. Equilibrium paths for perfect ($\bar{\xi} = 0$) and imperfect ($\bar{\xi} = 0.05$) system.

paths $\mathscr{S}_i$ of the symmetric system. In addition, a secondary bifurcation point manifests itself on the intersection of curves $\mathscr{B}_1$ and $\mathscr{B}_4$. This point cannot obviously be determined by the standard method, unless secondary bifurcations are searched explicitly along all bifurcated paths. Far from bifurcation the equilibrium curve $\mathscr{B}_1$ also approaches a path of the symmetric system.

Distances between parallel paths are found to be proportional to χ so that, as the asymmetry approaches zero, paths $\mathscr{B}$ and $\mathscr{S}$ and the bifurcation points tend to coincide. This clarifies why three equilibrium curves of the asymmetric system apparently split into four.

In conclusion, however small the asymmetry parameter χ is, provided it is different from zero, the behavior around the bifurcation point is of the asymmetric type but it changes rapidly approaching that of the symmetric system. This explains why extrapolations from the origin are not effective.

A comprehensive picture of the perfect and imperfect system is shown in Fig. 7 where both standard and modified perturbation methods have been employed for comparison. Curves $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$ corresponding to the perfect system have been plotted. In addition, paths corresponding to some particular shapes of initial imperfections with amplitude $\bar{\xi} = \sqrt{\bar{\varphi}^2 + \bar{\vartheta}^2} = 0.05$ are shown. These paths have been obtained in closed form in the SPM method and by solving the relevant equations through the Newton–Raphson procedure in the MPM method. It is apparent that the smaller the value of χ the more rapidly the two families of MPM and SPM curves diverge.

4. AN ALTERNATIVE GALERKIN PERTURBATION APPROACH

The modified perturbation method has served to overcome problems arising when the standard perturbation technique is employed to solve nearly symmetric systems; however, as already stated in Section 2.2, it can only be applied to a particular class of structures.

For this reason, it is more convenient to adopt an alternative approach by following the steps outlined in Byskov and Hutchinson (1979) and Sridharan and Benito (1984). This method, which in many respects is similar to the Koiter original formulation, is more general in that it allows small differences among the buckling loads to be taken into account and, besides, it does not present the drawbacks of the standard method.

The idea of the Galerkin perturbation method (GPM) consists of expressing the displacements field in the form

$$u = \xi_i v_i + \tfrac{1}{2}\xi_i\xi_j v_{ij} + \tfrac{1}{6}\xi_i\xi_j\xi_k v_{ijk} + \cdots + (\lambda - \lambda_c)[\tfrac{1}{2}\xi_i v_{i\lambda} + \tfrac{1}{6}\xi_i\xi_j v_{ij\lambda} + \cdots] + \cdots, \tag{30}$$

where ξ_i are the amplitudes of the m interacting buckling modes v_i and v_{ij}, v_{ijk} are displacements fields which are determined by solving appropriate perturbation equations as specified later. Through a Galerkin approach, the following equilibrium equations are obtained:

$$\left(1 - \frac{\lambda}{\lambda_{ci}}\right)\xi_i + a_{ijk}\xi_j\xi_k + a_{ijkl}\xi_j\xi_k\xi_l + \cdots = \bar{\xi}_i \quad \text{(no sum with respect to } i\text{)}, \tag{31}$$

where λ_{ci} $(i = 1, 2, \ldots, m)$ is the critical load associated with the buckling mode v_i. Equations (31) can be solved numerically to construct the $(\lambda, \xi_i; \bar{\xi}_i)$ path corresponding to a given initial imperfection $\bar{u}$ assumed as a linear combination $\bar{u} = \bar{\xi}_i v_i$.

Note that, in contrast with the standard Galerkin method, the displacements field (30) is not expressed as a linear combination of known functions. Indeed, the kinematical description takes into account the evolution of the structure deformation under increasing load in the post-critical range by means of the displacement fields v_{ij}, $v_{ijk}, \ldots$ In addition, in the GPM the equilibrium paths are determined by solving the non-linear equations which collect the quadratic and cubic terms all together, thus removing problems pointed out in Section 3 arising in the analysis of nearly-symmetric structures with the SPM.

Differences and analogies between the GPM and the SPM become apparent on the basis of the following considerations, relative to the case $\lambda_{ci} = \lambda_c$ $(i = 1, \ldots, m)$. To make our comparative analysis more transparent we start in the two approaches from the same series expansion (2), by temporarily setting aside eqn (30).

In the SPM eqn (5) is satisfied by forcing each term in ξ, $\xi^2, \ldots$ to vanish separately for any kinematically admissible δu; from this perturbation equations (6) are obtained. By applying the Fredholm condition to each perturbation equation, a relation is established between coefficients of the same order of the load and displacement series expansion; for instance eqn (9) forges a relationship between λ_1 and u_1, eqn (11) between λ_2 and u_2, through the arbitrary constants μ_i and β_i, respectively.

In the GPM no such relations are established and eqns (6) are solved for *arbitrary values of the load.* Due to the singularity of the operator Φ_c'', the solvability of the equations is ensured by introducing a constraint on the displacements field δu, that is by solving eqns (6) in a subspace of the kinematically admissible functions. By insisting, for instance, that u_2, $u_3, \ldots$ be orthogonal to each buckling mode v_k through a positive definite bilinear operator T, $Tv_k u_2 = 0, \ldots$, it must be in eqns (6) $Tv_k\delta u = 0$. We now observe that, in virtue of eqn (8), eqn (6b) admits the solution

$$u_2 = \mu_i\mu_j v_{ij} + \lambda_1\mu_i v_{i\lambda}, \tag{32}$$

where v_{ij} and $v_{i\lambda}$ are solutions of the variational problems

$$\begin{aligned} \Phi_c'' v_{ij}\delta u &= -\Phi_c''' v_i v_j \delta u \\ \Phi_c'' v_{i\lambda}\delta u &= -2\dot{\Phi}_c'' v_i \delta u, \end{aligned} \tag{33}$$

under the conditions

$$T v_k \delta u = 0 \quad (k = 1, 2, \ldots, m). \tag{34}$$

Equations (33) and (34) can be solved through a Lagrange multiplier technique. In contrast with the SPM, the coefficient λ_1 in eqn (32) is undetermined.

It is worth noticing that in this approach the orthogonality condition plays a fundamental role in that it ensures the solvability of the problem. On the contrary, in the Budiansky procedure, it serves as a normalization condition only.

At this stage of the procedure, if one substitutes eqn (8) for u_1, eqn (32) for u_2 etc. into (2a) the series expansion (30) is obtained, if one poses $\xi\mu_i = \xi_i$ and takes from eqn (2b) $\lambda_1\xi = \lambda - \lambda_c$.

By introducing u_1, u_2 etc. into eqn (5), the equilibrium equation is satisfied for any δu orthogonal to v_k $(i = 1, \ldots, m)$. By requesting the equation to be satisfied for $\delta u = v_k$ also, the equilibrium equations (31) are finally obtained in the modes with amplitudes $\xi_k = \xi\mu_k$ and in the load parameter $\lambda - \lambda_c = \lambda_1\xi + \cdots$. The last step of the method *formally* coincides with a procedure of the Galerkin type where the function u is expressed through eqn (30) and the test function δu as a variation of the *terms linear in* ξ *only*, $\delta u = v_k \delta\xi_k$. Indeed, higher order terms in ξ appearing in δu do not play any role in eqn (5), due to the particular choice of u_2, u_3.

Finally, we want to show how the GPM can advantageously be utilized in the analysis of nearly symmetric systems. The basic idea consists of expressing all quantities as a perturbation of those of an arbitrarily chosen symmetric system through a series expansion in terms of the asymmetry parameter χ. It is obviously convenient to choose the symmetric system in such a way as to make the solution of the problem as simple as possible. By proceeding in this way the total potential energy and the displacements field are written as

$$\begin{aligned} \Phi[u;\lambda;\chi] &= \Phi_0[u;\lambda] + \chi\tilde{\Phi}[u;\lambda] + O(\chi^2) \\ u(\chi) &= u^0 + \chi\tilde{u} + O(\chi^2), \end{aligned} \tag{35}$$

where $\Phi_0 = \Phi(u, \lambda; 0)$, $u^0 = u(0)$, $\tilde{\Phi} = (\partial\Phi/\partial\chi)_{\chi=0}$, $\tilde{u} = (\partial u/\partial\chi)_{\chi=0}$.

The eigenvalue problem (6a) now furnishes the solution

$$u_1 = \mu_i(v_i^0 + \chi\tilde{v}_i), \tag{36}$$

where v_i^0 are m simultaneous buckling modes of the symmetric system $\Phi''_{0c} v_i^0 \delta u = 0$ and the corrections $\tilde{v}_i$ are solutions of the problem

$$\Phi''_{0c}\tilde{v}_i\delta u = -\tilde{\Phi}''_c v_i^0 \delta u, \tag{37}$$

under the auxiliary conditions

$$T v_k^0 \delta u = 0 \quad (k = 1, 2, \ldots, m). \tag{38}$$

The second order displacements field is now

$$u_2 = \mu_i\mu_j(v_{ij}^0 + \chi\tilde{v}_{ij}) + \lambda_1\mu_i(v_{i\lambda}^0 + \chi\tilde{v}_{i\lambda}), \tag{39}$$

where v_{ij}^0 and $v_{i\lambda}^0$ are the secondary modes of the symmetric system which satisfy eqns (33), (34) whereas the corrections $\tilde{v}_{ij}$ and $\tilde{v}_{i\lambda}$ are determined by solving the variational problems

$$\begin{aligned} \Phi''_{0c}\tilde{v}_{ij}\delta u &= -2[\tilde{\Phi}'''_c v_i^0 v_j^0 + \Phi'''_{0c}(v_i^0\tilde{v}_j + v_j^0\tilde{v}_i)]\delta u \\ \Phi''_{0c}\tilde{v}_{i\lambda}\delta u &= -2[\dot{\tilde{\Phi}}''_c v_i^0 + \dot{\Phi}''_{0c}\tilde{v}_i]\delta u, \end{aligned} \tag{40}$$

under the constraints (38).

In conclusion, once the displacements of the symmetric system have been determined, the corresponding corrections which characterize the nearly symmetric system are evaluated by solving a sequence of linear problems where the stiffness operator is the simpler one of the symmetric structure. Then the non-linear equilibrium equations as previously illustrated are derived.

5. CONCLUSIONS

The non-linear interaction between several buckling modes in nearly symmetric structures has been investigated. The interest has been focused on a perturbation method capable of correctly describing the non-linear equilibrium paths of the perfect and imperfect structures. It has been shown that the standard perturbation method formulated by Budiansky fails whenever a small parameter χ which describes the asymmetry of the system approaches zero, thus preventing the recovery of the solution of the symmetric system. This drawback can be overcome for a restricted class of structures by shifting cubic terms which are responsible for the asymmetry behavior from second order to third order perturbation equations.

A system with two degrees of freedom has served to explain the problem and to show the role played by post-bifurcated paths in the description of the mechanical behavior of the structure.

An alternative method which follows the Galerkin approach has then been discussed by showing differences and analogies with the standard method. It is seen that the Galerkin method is more general and does not exhibit the drawbacks of the standard method in that the load–displacements law is determined by solving equations where all non-linearities of the problem appear at the same level. The method can then be specialized to analyze nearly symmetric structures by considering the actual system as a perturbation of a properly chosen symmetric system. In this way, the displacement fields are obtained by solving linear equations where the stiffness operator is the simpler one of the symmetric system.

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